

Increasing entanglement through engineered disorder in the random Ising chain

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The ground state entanglement entropy between block of sites in the random Ising chain is studied by means of the Von Neumann entropy. We show that in presence of strong correlations between the disordered couplings and local magnetic fields the entanglement increases and becomes larger than in the ordered case. The different behavior with respect to the uncorrelated disordered model is due to the drastic change of the ground state properties. The same result holds also for the random 3-state quantum Potts model.

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Entanglement plays a central role in modern quantum mechanics. Having been regarded for long time as the root of the incompleteness of quantum mechanics and the source of several paradoxes [1], with the advent of Quantum Information (QI) theory it has received renewed attention, and acquired the status of a fundamental resource for QI processing [2]. In the context of strongly correlated quantum systems entanglement arises in a natural way [3], and can in fact be regarded as a conceptual bridge between condensed matter physics and QI theory. In particular, much attention has been recently devoted to the ground state entanglement between two blocks of spins in one dimensional spin chains [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. By means of both analytical proofs and numerical simulations, this quantity has been shown to be logarithmically divergent with the length of the block when the spin chain is critical. Remarkably, the prefactor of the logarithm is proportional to the central charge of the corresponding conformal field theory (CFT) associated with the spin model at hands [16].

The average block entropy has been also studied in the context of random critical spin chains [12, 17, 18, 19, 20, 21, 22]. In this case, on the one hand works on random Ising and XXZ spin 1/2 chains [12, 17, 18] suggest that the block entropy still grows logarithmically with the size of the block but with a smaller prefactor of the logarithm with respect to the ordered model, being “renormalized” by a factor $\ln 2$. On the other hand, the block entropy in random quantum Potts chains with spin dimension $d \geq 2$ has been studied in [19], where a strong-disorder renormalization group (RG) analysis shows that the average entropy still diverges logarithmically but that the prefactor of the entropy is not proportional to the one of the

corresponding pure model and becomes larger than the latter when $d > 41$. This provides evidence that disorder can increase entanglement with respect to pure models, contrary to naïve expectations. The same analysis have been performed in aperiodic spin 1/2 chains and again it has been shown that the prefactor of the entropy logarithmic scaling is not simply the “renormalized” factor of the homogeneous model [23]. Therefore at present the possibility of associating an effective central charge to a non-homogeneous system is unclear.

In this work we study a random critical Ising spin chain [24] where the (nearest neighbor) couplings and the local transverse magnetic field are drawn from the same probability distribution and in addition share a certain (tunable) degree of correlation. We compute the average block entropy for this model and show that the prefactor of the entropy in a class of random correlated chain is *larger* than in the homogeneous case. An analogous result is found to hold also for larger dimension as we demonstrate in the random correlated quantum Potts model with $d = 3$.

We consider a spin chain with open boundary conditions and with d states $|0\rangle, |1\rangle, \dots, |d-1\rangle$ per lattice site, described by the Hamiltonian [25]

$$\mathcal{H}_d = - \sum_{i=1}^{L-1} J_i \sum_{n=1}^{d-1} (\bar{S}_i^z S_{i+1}^z)^n - \sum_{i=1}^L h_i \sum_{n=1}^{d-1} \Gamma_i^n. \quad (1)$$

Here L is the length of the chain, Γ represents the ladder operator $\Gamma|s\rangle = |(s+1) \bmod d\rangle$, $\langle k|S^z|k'\rangle = e^{2i\pi k/d} \delta_{kk'}$ ($k, k' = 0, \dots, d-1$) and \bar{S}^z is the hermitian conjugate of S^z .

For $d = 2$, $S^z \equiv \sigma^z$ (with $\sigma^{(x,y,z)}$ the Pauli matrices) and the model reduces to the (random) transverse field Ising model

$$\mathcal{H}_2 = - \sum_{i=1}^{L-1} J_i \sigma_i^z \sigma_{i+1}^z - \sum_{i=1}^L h_i \sigma_i^x. \quad (2)$$

In the pure case $J_i = h_i = 1$ while in the random case J_i and h_i are positive random numbers drawn from a joint

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distribution $\mathcal{P}_\alpha(J, h)$ defined in the interval $[0, 1]^{\times 2}$ such that the marginal probability distributions

$$P(J) = \int_0^1 dh \mathcal{P}_\alpha(J, h) \quad (3)$$

$$P(h) = \int_0^1 dJ \mathcal{P}_\alpha(J, h) \quad (4)$$

are both uniform distributions in the interval $[0, 1]$. This ensures that the random models we consider are critical [24]. The correlation coefficient characterizing the joint distribution \mathcal{P}_α is defined as $\alpha = \langle (J - \langle J \rangle)(h - \langle h \rangle) \rangle / (\sigma_J \sigma_h)$ where σ_J (σ_h) is the standard deviation of J (h). In the following we will consider the interval $0 \leq \alpha \leq 1$ whose endpoints correspond to independent and perfectly correlated couplings and transverse fields, respectively.

We study the ground state entanglement between a block of the first ℓ spins and the rest of the chain. This is quantified by the von Neumann entropy of the reduced density matrix of the block:

$$S_\ell = -\text{Tr} \rho_\ell \log_2 \rho_\ell \quad (5)$$

$$\rho_\ell = \text{Tr}_{L-\ell} |\psi_G\rangle \langle \psi_G| \quad (6)$$

with $|\psi_G\rangle$ the ground state of the Hamiltonian. In the random case we average over N independent disorder realizations.

For homogeneous models, CFT calculations show that the entropy scales as [5]:

$$\mathcal{S}_\ell = \frac{\kappa}{6} \log_2 \left[\frac{L}{\pi} \sin \left(\frac{\pi}{L} \ell \right) \right] + A, \quad (7)$$

where κ is equal to the central charge c of the corresponding theory and A is a non-universal constant [5, 29]. The central charge corresponding to the homogenous quantum Potts model depends on the local site dimension d as

$$c_d^h = 2 \frac{d-1}{d+2}, \quad (8)$$

[19]. For general non-homogeneous models, as said before, the entropy still grows logarithmically but with a different prefactor $\kappa \neq c$ [17, 19, 20].

The $d = 2$ model can be solved analytically in both the homogeneous and disordered case as it reduces, as noticed earlier, to the transverse field Ising model [26, 27]: one first performs a Jordan-Wigner transformation that maps the spins into fermions, then using a Bogoliubov transformation, whose coefficients are found via an exact diagonalization, the system is mapped into non-interacting fermions. Finally, the eigenvalues of the reduced density matrix ρ_ℓ and thus the entanglement can be determined from the two-point regular and anomalous correlation functions as shown in [28].

For systems with higher dimensionality one cannot resort on any analytical tools, and in order to study the

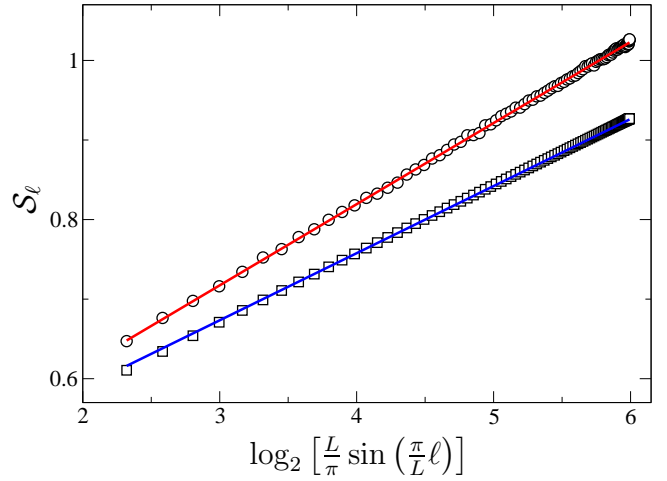


FIG. 1: (color online) Block entropy (in bits) averaged over $N = 5 \times 10^3$ configurations for an Ising spin chain of $L = 200$ sites in the homogeneous (squares) and perfectly correlated random (circles) cases. Best-fit lines with expression (7) are also shown.

3—states quantum Potts model we turn to the finite size density matrix renormalization group (DMRG) method [30, 31, 32]. The strategy behind the DMRG algorithm is to construct a system block and then recursively enlarge it, until the desired system size is reached. At every step the basis of the corresponding Hamiltonian is truncated, so that the size of the Hilbert space is kept manageable as the physical system grows. The truncation of the Hilbert space is performed by retaining the eigenstates corresponding to the m highest eigenvalues of the block reduced density matrix. Several sweeps of the finite system DMRG are performed to increase the accuracy for non-homogeneous chains.

d=2 results: Ising model – In Fig.1 we show the results for the block entropy evaluated for the homogenous and perfectly correlated random ($\alpha = 1$, $N = 5 \times 10^3$ configurations) Ising chains of total length $L = 200$. Fitting the numerical data with expression (7) we obtain for the homogeneous system $\kappa_2^h = 0.50 \pm 0.01$, in perfect agreement with Eq.(8) which gives $c_2^h = 0.5$ for the homogeneous chain with $d = 2$. We obtain $\kappa_2^{\alpha=1} = 0.61 \pm 0.01$ for the perfectly correlated $\alpha = 1$ case: Notice that, differently from previous studies on random spin one half chains, $\kappa_2^{\alpha=1}$ is *larger* than the prefactor for the homogeneous case resulting (in the thermodynamic limit) in an increment of the entanglement in the disordered chain with respect to the homogeneous case [33].

An important question that ought to be studied is the stability of this behavior with the correlation α shared by the couplings J_i and the transverse fields h_i . In Fig.2 we show the prefactor κ_2^α of the average entropy for different chain lengths ($L = 50, 70, 100$) as a function of the correlation α . The results show that for small α we confirm the analytic result of [17] apart from finite size errors. Indeed, a finite size scaling (see inset of Fig.2) gives $\kappa_2^{\alpha=0} = 0.30 \pm 0.04$ quite in agreement with the

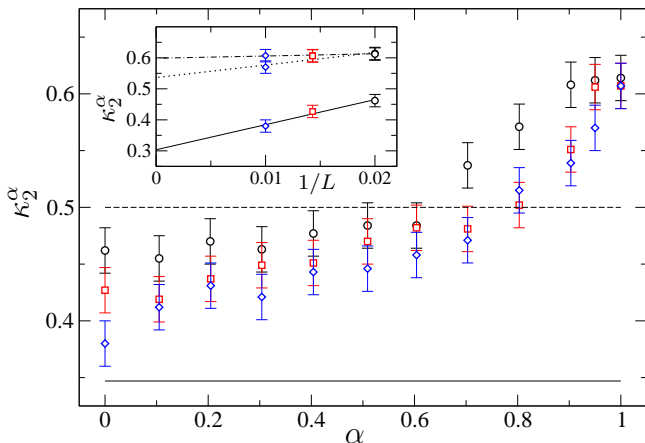


FIG. 2: (color online) Prefactor κ_2^α of the entropy versus the correlation coefficient α of the J and h couplings for different chain lengths ($L = 50$, black circles; $L = 70$, red squares; $L = 100$, blue diamonds) and $N = 10^4$ configurations. The solid and dashed line gives $\kappa_2^{\alpha=0} = c_2 \ln 2 \approx 0.35$ and $\kappa_2^h = c_2 = 0.5$ respectively. The inset shows the finite-size scaling of the prefactor with best-fit lines (solid, $\alpha = 0$; dotted, $\alpha = 0.95$; dashed-dotted, $\alpha = 1$). Extrapolation to the thermodynamic limit $1/L \rightarrow 0$ gives $\kappa_2^{\alpha=0} = 0.30 \pm 0.04$ and $\kappa_2^{\alpha=1} = 0.60 \pm 0.01$.

expected result $\kappa_2^{\alpha=0} = c_2^h \ln 2 \approx 0.346$; in the perfect correlated case one gets instead $\kappa_2^{\alpha=1} = 0.60 \pm 0.01$ in the thermodynamic limit. For intermediate values of α our results show the departure from the completely correlated random Ising model and for strong enough correlations ($\alpha > \alpha^* \sim 0.9$) the prefactor κ_2^α become larger than the prefactor for the pure model $\kappa_2^h = 0.5$. This result is a clear evidence that it is possible to increase the entanglement adding disorder to an homogenous system already for small local dimension $d = 2$. Notice that the scaling of κ_2^α with the system size is faster for stronger correlations. Indeed, the perfectly correlated case, $\alpha = 1$, seems at convergence already with $L = 50$ while the uncorrelated case is still far for the thermodynamic limit at $L = 100$. This clearly reflects the largest space of the possible configurations of the uncorrelated case. However, the finite size scaling suggests that in the whole range $\alpha^* < \alpha < 1$ we have $\kappa_2^\alpha > \kappa_2^h = c_2$ in the thermodynamical limit.

A deeper insight on the ways the $\alpha > \alpha^*$ case differs from the $\alpha = 0$ one, can be gained by determining the probability distribution P of the block entropy for a fixed block length in the two cases. This procedure will in fact unravel the presence of long distance effective quantum correlations in the chain ground state. The resulting distributions for $L = 100$, $\ell = 50$ and $N = 5 \times 10^3$ are shown in Fig. 3. From these results it emerges that the ground states for the two models are structurally different: for $\alpha = 0$ a single narrow peak at $S_{50} = 1$ denotes the signature of the random singlet-like phase [17, 18]; for $\alpha = 1$ the peak at $S_{50} = 1$ is wider indicating the occurrence of a more complex structure. This is a clear signature that the clustering procedure described in [17] breaks

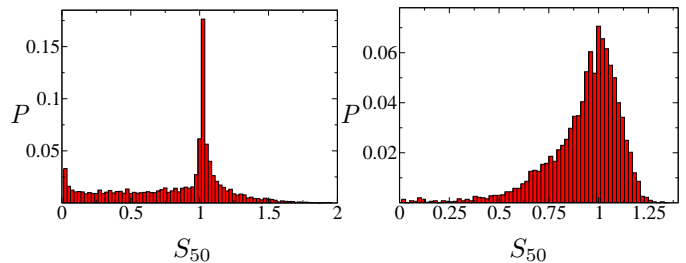


FIG. 3: (color online) Probability distribution for the von Neumann block entropy for a block of length $\ell = 50$ of an $L = 100$ chain for $N = 5 \times 10^3$ configurations. Left panel: $\alpha = 0$ (random uncorrelated case); right panel: $\alpha = 1$ (random perfectly correlated case.)

down. Indeed, the strong disorder renormalization procedure rely on the renormalization of the strongest bonds or local field, obtaining either a ferromagnetic cluster or a frozen spin. However, if the magnetic field and the coupling are always of the same order this renormalization procedure may not be applied. It seems there exists a crossover between the two extreme regimes, namely, the random-singlet phase and the delocalized one (see Fig. 3).

d=3 results: Potts model – In Fig.4 we show the block entropy for $L = 70$ in the homogeneous and perfectly correlated random cases for $d = 3$, evaluated through a finite size DMRG algorithm using $m = 30$ and $N = 3 \times 10^3$ configurations. With these parameters the maximum error in the entropy of a single configuration is less than 5×10^{-3} . The fit of the data for the pure model according to expression (7) gives $\kappa_3^h = 0.83 \pm 0.02$, in good agreement with value predicted by the CFT formula of Eq.(8) $c_3 = 4/5 = 0.8$. In the disordered case we find a similar picture as for the $d = 2$ case: in the perfectly correlated random case we obtain a bigger value of the prefactor $\kappa_3^{\alpha=1} = 0.91 \pm 0.02 > \kappa_3^h$. For $\alpha = 0$ it is lower than that for the homogeneous model $\kappa_3^{\alpha=0} = \ln 3/2 \approx 0.55$ (computed in [19] via the strong disorder renormalization group).

Conclusions – In this paper we have considered a class of random critical spin chains described by the Hamiltonian of Eq.(1). We concentrated on two models: the random Ising model with $d = 2$ and the quantum Potts model with $d = 3$: the couplings J and the local transverse magnetic fields h are random but share a given amount of correlations. Our analysis shows that whenever J and h are correlated the prefactor of the Von Neumann entropy is always greater than the prefactor for the uncorrelated case. Moreover, when the correlations are above a certain threshold α^* the prefactor of the random correlated model is larger than the prefactor for the homogenous chain. It would be interesting to confirm our numerical results applying a strong disorder renormalization procedure. In this work we first demonstrate how to increase entanglement by putting static disorder in systems with small local dimension. It is not needed to

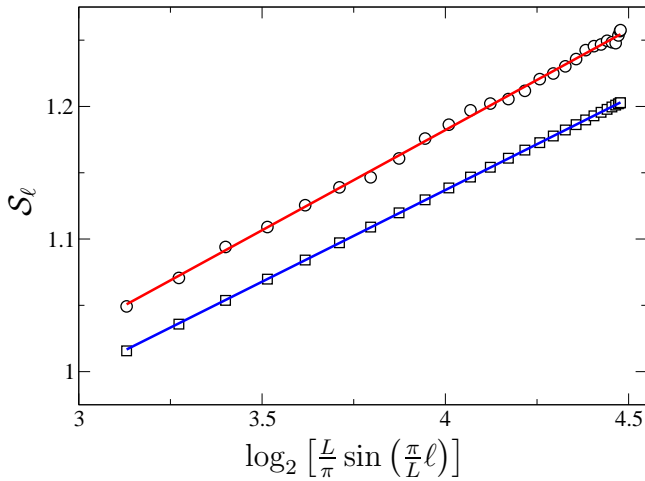


FIG. 4: (color online) Block entropy (in bits) averaged over $N = 3 \times 10^3$ configurations for a $d = 3$ Potts model of $L = 70$ sites in the homogeneous (squares) and perfectly correlated random (circles) cases. Best-fit lines with expression (7) are also shown.

consider systems with high local dimensions $d > 41$ as in [19]: our results indeed shows that exists a class of disordered spin one-half models where the ground state entanglement is larger than in the correspondent translational-

invariant ones. Notice that this random correlated model could be realized experimentally in engineered quantum systems, e.g. in optical lattices, ion traps or arrays of Josephson junctions. We conclude observing that the correlated random models studied in this work lay outside the random singlet-like phase for which a generalized c-theorem holds. It is thus ambiguous the association of a renormalized central charge to general non-homogeneous systems.

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